JOURNAL OF APPROXIMATION THEORY 18, 241–250 (1976)

Nonlinear Chebyshev Approximation Subject to Constraints

D. O. ANDREASSEN AND G. A. WATSON

Department of Mathematics, University of Dundee, Dundee, Scotland

Communicated by E. W. Cheney

Received May 28, 1975

1. INTRODUCTION

The theoretical and practical aspects of Chebyshev approximation problems where the unknown parameters are required to satisfy additional inequality or equality constraints have received a great deal of attention in recent years. Surveys of the work of various authors on this class of problem have been given by Taylor [10] and Lewis [9]. Characterization theorems for fairly general linear problems of this type are given by Laurent [8] and Andreassen [1], while certain classes of nonlinear problems have been treated by Hoffman [6, 7] and Gislason and Loeb [5].

It is the purpose of this paper to investigate the extent to which the characterization results for the general linear case can be extended to the nonlinear case, while imposing a minimum of restrictions on the problem. Necessary conditions and sufficient conditions of "zero in the convex hull" type are presented for local best approximations, as defined below. The theorems also generalize similar results for the nonlinear problem without constraints (see, for example, [11]). We remark that, although the results are fomulated for approximation in a finite interval $[a, b]$ of the single real variable x, no use is made of this restriction in the proofs, and hence the theorems are valid for multivariate approximation.

Let f, $\phi(-, \boldsymbol{\alpha}) \in C[a, b]$ be given functions, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)^T$, and let $\Omega \subset \mathbb{R}^n$ be given. Then the basic approximation problem with which we will be concerned can be stated: find $\boldsymbol{\alpha} \in \Omega$ to minimize

$$\| r(\boldsymbol{\alpha}) \| = \max_{x \in [a,b]} | r(x, \boldsymbol{\alpha}) | ,$$

where $r(-, \boldsymbol{\alpha}) = f - \phi(-, \boldsymbol{\alpha})$ and ϕ is nonlinear in the components of $\boldsymbol{\alpha}$. (For convenience we will often suppress the parameter $\boldsymbol{\alpha}$ in r, ϕ and similar expressions.)

Ω is said to be the set of feasible approximations. We will assume that Ω is nonempty and if there exists at least one $\boldsymbol{\alpha} \in \Omega$ which satisfies

$$\| r(\boldsymbol{\alpha}) \| \leqslant \| r(\boldsymbol{\beta}) \| \qquad (1.1)$$

for all $\boldsymbol{\beta} \in \Omega$, such an $\boldsymbol{\alpha}$ is a (global) best approximation. Because of the nonlinearity of ϕ as a function of $\boldsymbol{\alpha}$, it is possible for vectors $\boldsymbol{\alpha}$ to exist which satisfy (1.1) for all $\boldsymbol{\beta} \in \Omega \cap N(\boldsymbol{\alpha})$, where $N(\boldsymbol{\alpha})$ is some open neighborhood of $\boldsymbol{\alpha}$. In this case $\boldsymbol{\alpha}$ satisfies the usual (theoretical) definition of a local best approximation.

In the next two sections, we give necessary conditions for $\boldsymbol{\alpha}$ to be a local best approximation when Ω is defined by a set of inequality or a set of equality constraints. In addition, we examine the extent to which these conditions might also be sufficient, and show that sufficiency results of this form can be obtained provided that we permit a weakening of the definition of a local minimum. This is made precise by the next two definitions.

DEFINITION 1. C is an open cone of descent directions from $\boldsymbol{\alpha} \in \Omega$ if C is an open cone in $\mathbb{R}^k$, $k \leqslant n$, and there exists a vector-valued function $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)^T$, defined and continuously differentiable on some open neighborhood N of the origin in $\mathbb{R}^k$, with $\psi_{i+n-k}(\boldsymbol{\beta}) = \beta_i$, $i = 1, 2, \ldots, k$ (after rearranging the components if necessary), such that

(i) $\boldsymbol{\psi}(0) = 0$,

(ii) $\{\boldsymbol{\psi}(\boldsymbol{\beta}) \mid \boldsymbol{\beta} \in N\} \cap (\Omega - \boldsymbol{\alpha})$ is open relative to $\Omega - \boldsymbol{\alpha}$,

(iii) $\| r(\boldsymbol{\alpha} + \boldsymbol{\psi}(h\boldsymbol{\beta}))\| < \| r(\boldsymbol{\alpha})\|$, $\boldsymbol{\alpha} + \boldsymbol{\psi}(h\boldsymbol{\beta}) \in \Omega$,

for $\boldsymbol{\beta} \in C$ and $h > 0$ sufficiently small.

DEFINITION 2. $\boldsymbol{\alpha} \in \Omega$ is said to be a weak local best approximation if there exists no open cone of descent directions from $\boldsymbol{\alpha}$.

Remark 1. In the inequality constraint case we will restrict Ω to be convex with a nonempty interior, and then in Definition 1 $k = n$; i.e., C is an open cone in $\mathbb{R}^n$. In the equality constraint case, however, an element of Ω will have $<n$ degrees of freedom, hence the necessity of using a mapping from a cone in a lower-dimensional space.

Remark 2. In certain special cases (for example, when ϕ is a rational function), it is possible to show that $\boldsymbol{\alpha}$ is a local best approximation if and only if $\boldsymbol{\alpha}$ is a weak local best approximation.

The assumption that $(\partial/\partial\alpha_i)\, \phi$, $i = 1, 2, \ldots, n$, exist and are continuous as functions of x is central to the rest of this paper. We introduce the following notation:

$$\mathbf{v}_0(x, \boldsymbol{\alpha}) = [(\partial/\partial\alpha_1)\,\phi(x, \boldsymbol{\alpha}),..., (\partial/\partial\alpha_n)\,\phi(x, \boldsymbol{\alpha})]^T,$$

$$B_0(\boldsymbol{\alpha}) = \{x \mid x \in [a, b], \mid r(x, \boldsymbol{\alpha})\mid = \| r(\boldsymbol{\alpha})\|\},$$

and $S_0(\boldsymbol{\alpha})$ is a matrix each row of which is $\mathbf{v}_0{}^T(x, \boldsymbol{\alpha})$ for some $x \in B_0(\boldsymbol{\alpha})$. If $x_i \in B_0$, then $\theta_i = \operatorname{sign}(r(x_i, \boldsymbol{\alpha}))$, otherwise $\theta_i = 1$, and if $x_1,..., x_t$ are the points of B_0 present in S_0, then $D = \operatorname{diag}\{\theta_1,..., \theta_t\}$.

2. The Inequality Constraint Problem

Let p_j, $j = 1, 2,..., m$, be given functions of x and $\boldsymbol{\alpha}$, and let Ω be defined by

$$\Omega = \{\boldsymbol{\alpha} \mid p_j(x, \boldsymbol{\alpha}) \geqslant 0, j = 1, 2,..., m, x \in [a, b]\}.$$

We assume that $(\partial/\partial\alpha_i)\,p_j(-, \boldsymbol{\alpha})$ exists and is continuous as a function of x for all $\boldsymbol{\alpha} \in \Omega$, $i = 1, 2,..., n$, $j = 1, 2,..., m$.

In addition to the notation introduced earlier, we will in this section require the following. For a given $\boldsymbol{\alpha}$, define $\mathbf{v}_j(x, \boldsymbol{\alpha})$ to be an n-vector with ith component $(\partial/\partial\alpha_i)\,p_j(x, \boldsymbol{\alpha})$,

$$B_j(\boldsymbol{\alpha}) = \{x \mid x \in [a, b], p_j(x, \boldsymbol{\alpha}) = 0\},$$

$S_j(\boldsymbol{\alpha})$ to be a matrix each row of which is $\mathbf{v}_j{}^T(x, \boldsymbol{\alpha})$ for some $x \in B_j(\boldsymbol{\alpha})$, $j = 1, 2,..., m$, and

$$S(\boldsymbol{\alpha}) = \begin{bmatrix} S_0(\boldsymbol{\alpha}) \\ \vdots \\ S_m(\boldsymbol{\alpha}) \end{bmatrix}.$$

Theorem 1. *If $\boldsymbol{\alpha}$ is a local best approximation, and if there exists an open neighborhood of $\boldsymbol{\alpha}$ where $(\partial^2/\partial\alpha_i\,\partial\alpha_k)\,\phi$, $(\partial^2/\partial\alpha_i\,\partial\alpha_k)\,p_j$, $i, k = 1, 2,..., n$, $j = 1, 2,..., m$, exist and are uniformly bounded as functions of x, then there exists a set of $q \leqslant n + 1$ points $x_1,..., x_q \in \bigcup_{j=0}^m B_j$ and a nontrivial vector $\boldsymbol{\lambda}$ such that*

$$\boldsymbol{\lambda}^T S = 0, \qquad \lambda_i\theta_i \geqslant 0, \qquad i = 1, 2,..., q.$$

Proof. Suppose $\boldsymbol{\alpha}$ is a local best approximation, but no such set of points $x_1,..., x_q$ and no such vector $\boldsymbol{\lambda}$ exist.

Let $\bar{S}_i$ be the matrix with a row $\mathbf{v}_i{}^T(x)$ corresponding to *every* point of B_i. It follows from the theorem of Caratheodory [2, p. 17] that there does not exist any vector $\mu \geqslant 0$ with only a finite number of positive components such that

$$\mu^T \begin{bmatrix} \bar{D}\bar{S}_0 \\ \bar{S}_1 \\ \vdots \\ \bar{S}_m \end{bmatrix} = 0, \qquad \sum_i \mu_i = 1,$$

where $\bar{D}$ extends D in an obvious way.

Because of the continuity of $\mathbf{v}_j$ the subset of $\mathbb{R}^n$ consisting of the rows of $\bar{D}\bar{S}_0, \bar{S}_1, \ldots, \bar{S}_m$ is compact. Hence by a theorem on linear inequalities [2, p. 19], the nonexistence of μ implies the existence of a γ satisfying

$$\bar{D}\bar{S}_0\gamma > 0, \tag{2.1}$$

$$\bar{S}_j\gamma > 0, \qquad j = 1, 2, \ldots, m. \tag{2.2}$$

Equation (2.1) may be used to show that

$$\| r(\alpha + h\gamma)\| < \| r(\alpha)\|$$

for $h > 0$ and sufficiently small. The proof is given in [3].

Because of the existence and uniform boundedness of the second order partial derivatives in an open neighborhood of α, we can for sufficiently small h write

$$p_j(x, \alpha + h\gamma) = p_j(x, \alpha) + h\mathbf{v}_j^T(x, \alpha)\, \gamma + O(h^2),$$

where the bound on the $O(h^2)$-term is independent of x. Using this and the fact that $\mathbf{v}_j^T(x, \alpha)\, \gamma$ is bounded on $[a, b]$ and bounded away from zero for $x \in B_j$, it is easy to show that $p_j(x, \alpha + h\gamma) \geqslant 0$ for all $x \in [a, b]$ if $h > 0$ is sufficiently small.

We now have shown that for $h > 0$ sufficiently small, $\alpha + h\gamma$ is feasible and better than α. Since α is a local best approximation, this is a contradiction.

Remark. To prove that $\alpha + h\gamma$ is feasible and better than α, we only needed to know that (2.1) and (2.2) held. Since B_j is compact and $\mathbf{v}_j$ is continuous, we can easily show that (2.1) and (2.2) hold in an open neighborhood of γ. But then we have an open cone of descent directions from α (with $k = n$); i.e., Theorem 1 is also valid for weak local best approximations.

THEOREM 2. *Let $\alpha \in \Omega$ be given, and assume*

(i) *Ω is convex,*

(ii) *Ω has a nonempty interior.*

Then $\boldsymbol{\alpha} \in \Omega$ *is a weak local best approximation if there exists a set of* $q \leqslant n + 1$ *points* $x_1, \ldots, x_q \in \bigcup_{j=0}^{m} B_j$ *and a nontrivial vector* $\boldsymbol{\lambda}$ *such that*

$$\boldsymbol{\lambda}^T S = 0, \qquad \lambda_i \theta_i \geqslant 0, \qquad i = 1, 2, \ldots, q,$$

with no row of S identically zero.

Proof. Assume $x_1, \ldots, x_q$ and $\boldsymbol{\lambda}$ exist, but $\boldsymbol{\alpha}$ is not a weak local best approximation. Then there exists an open cone C of descent directions from $\boldsymbol{\alpha}$. As mentioned earlier, because of (i) and (ii) it is possible to prove that C is an open cone in $\mathbb{R}^n$. Hence we have that if $\boldsymbol{\gamma} \in C$ then

$$\boldsymbol{\alpha} + h\boldsymbol{\gamma} \in \Omega, \tag{2.3}$$

$$\| r(\boldsymbol{\alpha} + h\boldsymbol{\gamma})\| < \| r(\boldsymbol{\alpha})\| \tag{2.4}$$

if $h > 0$ is sufficiently small.

Since ϕ, p_j, $j = 1, 2, \ldots, m$, are differentiable with respect to the components of $\boldsymbol{\alpha}$, we get

$$r(x, \boldsymbol{\alpha} + h\boldsymbol{\gamma}) = r(x, \boldsymbol{\alpha}) - h \sum_{i=1}^{n} \gamma_i(\partial/\partial\alpha_i)\, \phi(x, \boldsymbol{\alpha}) + hO(h),$$

$$p_j(x, \boldsymbol{\alpha} + h\boldsymbol{\gamma}) = p_j(x, \boldsymbol{\alpha}) + h \sum_{i=1}^{n} \gamma_i(\partial/\partial\alpha_i)\, p_j(x, \boldsymbol{\alpha}) + hO(h).$$

Equations (2.3), (2.4), and the definition of D and S_j give us

$$DS_0\boldsymbol{\gamma} \geqslant 0, \tag{2.5}$$

$$S_j\boldsymbol{\gamma} \geqslant 0, \qquad j = 1, 2, \ldots, m. \tag{2.6}$$

Obviously these inequalities hold for all vectors $\boldsymbol{\gamma} \in C$. Since C is an open cone in $\mathbb{R}^n$, there exists a $T > 0$ such that $\boldsymbol{\gamma} + t\boldsymbol{\delta} \in C$ for all $t \in [0, T]$ and all unit vectors $\boldsymbol{\delta}$. Since no row of S is identically zero, it follows that for all $\boldsymbol{\gamma} \in C$ we have

$$DS_0\boldsymbol{\gamma} > 0 \tag{2.7}$$

$$S_j\boldsymbol{\gamma} > 0, \qquad j = 1, 2, \ldots, m. \tag{2.8}$$

This contradicts the existence of the nontrivial vector $\boldsymbol{\lambda}$.

Remark 1. We see that to prove (2.5) and (2.6) we only needed the existence of *one* descent direction, i.e., $\boldsymbol{\gamma}$. To get (2.7) and (2.8), however, we needed an open cone of descent directions. It is difficult to see how the present proof can be modified to avoid this.

Remark 2. It is clearly possible that the constraint functions p_j are given in such a way that there exists a functional relationship between the coordinates of Ω. This means that Ω has an empty interior relative to $\mathbb{R}^n$ which is equivalent to the existence of a linear relationship between the coordinates of Ω. (This is a simple corollary of [4, Theorem 4, p. 16].) Hence condition (ii) of Theorem 2 is necessary to ensure that the approximation problem really is n-dimensional.

In practice a verification of (ii) will be difficult, but the following (linear) example shows that (ii) is essential for Theorem 2. (See also [8].)

EXAMPLE. $f(x) = x^2$, $\phi(x, \boldsymbol{\alpha}) = \alpha_1 + \alpha_2 x$, $p_1(x, \boldsymbol{\alpha}) = \alpha_1 + (\alpha_2 - 1)\,x$, $p_2(x, \boldsymbol{\alpha}) = -\alpha_1 + \alpha_2(x - 1) + 1 + x$, $x \in [0, 1]$.

$\boldsymbol{\alpha} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is feasible and gives

$$DS_0 = [-1 \;\; 0], \qquad S_1 = -S_2 = [1 \;\; 1],$$

i.e., $\boldsymbol{\lambda}^T = [0 \;\; 1 \;\; 1]$ satisfies $\boldsymbol{\lambda}^T S = 0$, $\lambda_i \theta_i \geqslant 0$, $i = 1, 2, 3$.

$p_1(1, \boldsymbol{\alpha}) \geqslant 0$ and $p_2(0, \boldsymbol{\alpha}) \geqslant 0$ imply $\alpha_1 + \alpha_2 = 1$, hence $\Omega = \{\boldsymbol{\alpha} \mid \alpha_1 + \alpha_2 = 1, \alpha_1 \in [0, 2]\}$; i.e., Ω has no interior points relative to $\mathbb{R}^2$, and we cannot apply Theorem 1.

Using $\psi_1(x) = -x$ it is easily seen that $C = \{x \mid x > 0\}$ is an open cone of descent directions from $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

3. The Equality Constraint Problem

We will now assume that p_j is independent of x and defined on some open subset $E \subseteq \mathbb{R}^n$, and in addition that $p_j \in C^1(E)$, $j = 1, 2,..., m$.

We define Ω by

$$\Omega = \{\boldsymbol{\alpha} \mid p_j(\boldsymbol{\alpha}) = 0, \qquad j = 1, 2,..., m\},$$

and the matrix $A(\boldsymbol{\alpha})$ as the matrix with (j, i)-element

$$a_{ji} = (\partial/\partial\alpha_i)\, p_j(\boldsymbol{\alpha}), \qquad j = 1, 2,..., m, \qquad i = 1, 2,..., n.$$

In addition we will use the following notation.

If $\boldsymbol{\alpha} \in \mathbb{R}^n$, then $\tilde{\boldsymbol{\alpha}} = [\alpha_{m+1},..., \alpha_n]^T$, and if $\Gamma \subseteq \mathbb{R}^n$, then $\tilde{\Gamma} = \{\tilde{\boldsymbol{\gamma}} \mid \boldsymbol{\gamma} \in \Gamma\}$.

Now let $\boldsymbol{\alpha} \in \Omega$ be fixed and assume $A(\boldsymbol{\alpha})$ has rank m. Then we can, after renaming the components of $\boldsymbol{\alpha}$ if necessary, write $A = [B\ C]$, where B is $m \times m$ and nonsingular. Since $p_j \in C^1(E)$, $j = 1, 2,..., m$, then there exists a neighborhood $N \subseteq E$ of $\boldsymbol{\alpha}$ where B is nonsingular. The implicit function theorem gives us that there exists an open neighborhood $M \subseteq N$ of $\boldsymbol{\alpha}$ and functions $\psi_1, \ldots, \psi_m \in C^1(\tilde{M})$ such that $\alpha_j = \psi_j(\tilde{\boldsymbol{\alpha}}), j = 1, 2, \ldots, m$, and

$$\boldsymbol{\psi}(\tilde{\boldsymbol{\beta}}) = (\psi_1(\tilde{\boldsymbol{\beta}}), \ldots, \psi_m(\tilde{\boldsymbol{\beta}}), \tilde{\boldsymbol{\beta}}^T)^T \in \Omega \qquad \text{for} \quad \tilde{\boldsymbol{\beta}} \in \tilde{M}.$$

Let $\tilde{\phi}$ be the function defined by

$$\tilde{\phi}(-, \tilde{\boldsymbol{\beta}}) = \phi(-, \boldsymbol{\psi}(\tilde{\boldsymbol{\beta}})). \tag{3.1}$$

Consider, now, the unconstrained approximation problem: find $\tilde{\boldsymbol{\beta}} \in \tilde{M}$ to minimize

$$\|f - \tilde{\phi}(\tilde{\boldsymbol{\beta}})\|. \tag{3.2}$$

The following lemma shows that (3.2) and the original constrained problem are, in a certain sense, equivalent.

LEMMA 1. *$\tilde{\boldsymbol{\alpha}}$ is a weak local best approximation to the problem defined by* (3.2) *if and only if $\boldsymbol{\alpha}$ is a weak local best approximation to the original problem.*

Proof. Suppose $\boldsymbol{\alpha}$ is not a weak local best approximation to the original problem. Then there exists an open cone C of descent directions from $\boldsymbol{\alpha}$. Since the mapping from $\mathbb{R}^k$ into $\mathbb{R}^n$ of C must give a set whose intersection with $\Omega - \boldsymbol{\alpha}$ is open relative to $\Omega - \boldsymbol{\alpha}$, and $\tilde{M}$ is an open subset of $\tilde{\Omega}$, the mapping must be of the form $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)^T$, where $\tau_{n-k+i}(\tilde{\boldsymbol{\beta}}) = \beta_i$, $i = 1, 2, \ldots, k$, and $k \geqslant n - m$. But $\psi_1, \ldots, \psi_m$ defined by the implicit function theorem are unique, hence, with a suitable definition of ψ_i outside $\tilde{M}$, we have that $\boldsymbol{\tau} \equiv \boldsymbol{\psi}$; i.e., C is open relative to $\mathbb{R}^{n-m}$ and is an open cone of descent directions from $\tilde{\boldsymbol{\alpha}}$ for (3.2).

The proof of the second half of the lemma is similar and will not be given here.

Since (3.2) is a special case of the inequality constraint problem, we will use Theorems 1 and 2 to characterize a solution to (3.2) and hence, by Lemma 1, a solution to the equality constraint problem.

Differentiating (3.1) we get

$$\frac{\partial}{\partial \tilde{\beta}_j} \tilde{\phi} = \sum_{k=1}^{m} \frac{\partial \phi}{\partial \alpha_k} \frac{\partial \psi_k}{\partial \tilde{\beta}_j} + \frac{\partial \phi}{\partial \alpha_{m+j}}, \qquad j = 1, 2, \ldots, n - m. \tag{3.3}$$

Also, since

$$0 \equiv p_j(\psi_1(\tilde{\boldsymbol{\beta}}), \ldots, \psi_m(\tilde{\boldsymbol{\beta}}), \tilde{\boldsymbol{\beta}}),$$

we get

$$0 = \sum_{k=1}^{m} {}_i\frac{\partial p_j}{\partial \alpha_k} \frac{\partial \psi_k}{\partial \tilde{\beta}_l} + \frac{\partial p_j}{\partial \alpha_{m+l}}, \qquad j = 1, 2, \ldots, m, \; l = 1, 2, \ldots, n - m. \tag{3.4}$$

Let T, U, V, W be the matrices defined by

$$t_{ij} = (\partial\tilde{\phi}/\partial\tilde{\beta}_j)(x_i, \tilde{\boldsymbol{\alpha}}), \qquad i = 1, 2,..., q, \quad j = 1, 2,..., n - m,$$

$$u_{ij} = (\partial\phi/\partial\alpha_j)(x_i, \boldsymbol{\alpha}), \qquad i = 1, 2,..., q, \quad j = 1, 2,..., m,$$

$$v_{ij} = (\partial\phi/\partial\alpha_{m+j})(x_i, \boldsymbol{\alpha}), \qquad i = 1, 2,..., q, \quad j = 1, 2,..., n - m,$$

$$w_{ij} = (\partial\psi_i/\partial\tilde{\beta}_j)(\tilde{\boldsymbol{\alpha}}), \qquad i = 1, 2,..., m, \quad j = 1, 2,..., n - m.$$

Now (3.3) and (3.4) give

$$t_{ij} = \sum_{k=1}^{m} u_{ik}w_{kj} + v_{ij}, \qquad i = 1, 2,..., q, j = 1, 2,..., n - m,$$

$$0 = \sum_{k=1}^{m} b_{ik}w_{kj} + c_{ij}, \qquad i = 1, 2,..., m, j = 1, 2,..., n - m,$$

or

$$T = UW + V, \qquad 0 = BW + C.$$

Since B is nonsingular we can eliminate W to get

$$T = V - UB^{-1}C. \tag{3.5}$$

Suppose now that there exists a nontrivial $\boldsymbol{\gamma}$ such that

$$\boldsymbol{\gamma}^T T = 0, \qquad \gamma_i\theta_i \geqslant 0, \qquad i = 1, 2,..., q. \tag{3.6}$$

Using (3.5) we get

$$\boldsymbol{\gamma}^T V = \boldsymbol{\gamma}^T UB^{-1}C,$$

and defining $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu}^T = -\boldsymbol{\gamma}^T UB^{-1},$$

we get

$$\boldsymbol{\gamma}^T V + \boldsymbol{\mu}^T C = 0, \qquad \boldsymbol{\gamma}^T U + \boldsymbol{\mu}^T B = 0.$$

Since $[U \; V] = S_0$ and $[B \; C] = A$, we can define $\boldsymbol{\lambda}^T = [\boldsymbol{\gamma}^T \; \boldsymbol{\mu}^T]$ to get

$$\boldsymbol{\lambda}^T \begin{bmatrix} S_0 \\ A \end{bmatrix} = 0, \qquad \lambda_i\theta_i \geqslant 0, \qquad i = 1, 2,..., q. \tag{3.7}$$

Clearly we also have that (3.7) implies (3.6). Since T is the matrix S_0 obtained by applying Theorem 1 or 2 to the problem (3.2), we have proved the following two theorems.

THEOREM 3. *Let* $\boldsymbol{\alpha}$ *be a local best approximation to the equality constraint problem, and assume that*

(i) $A(\boldsymbol{\alpha})$ *has full rank,*

(ii) *there exists an open neighborhood* $\subset E$ *of* $\boldsymbol{\alpha}$ *where* $(\partial^2/\partial\alpha_i\,\partial\alpha_k)\,\phi$, $(\partial^2/\partial\alpha_i\,\partial\alpha_k)\,p_j$, $i,\ k = 1, 2,..., n,\ j = 1, 2,..., m$, *exist and are uniformly bounded as functions of* x.

Then there exist a set of $q \leqslant n + 1 - m$ *points* $x_1,..., x_q \in B_0$ *and a nontrivial vector* $\boldsymbol{\lambda}$ *such that*

$$\boldsymbol{\lambda}^T \begin{bmatrix} S_0 \\ A \end{bmatrix} = 0, \qquad \lambda_i\theta_i \geqslant 0, \qquad i = 1, 2,..., q.$$

THEOREM 4. $\boldsymbol{\alpha} \in \Omega$ *is a weak local best approximation to the equality constraint problem if there exist a set of* $q \leqslant n + 1 - m$ *points* $x_1,..., x_q \in B_0$ *and a nontrivial vector* $\boldsymbol{\lambda}$ *such that*

$$A(\boldsymbol{\alpha}) \text{ has full rank,}$$

$$\boldsymbol{\lambda}^T \begin{bmatrix} S_0 \\ A \end{bmatrix} = 0, \qquad \lambda_i\theta_i \geqslant 0, \qquad i = 1, 2,..., q,$$

$$V - UB^{-1}C \text{ has no row identically zero.}$$

Remark. Using the same method we can prove theorems similar to Theorems 3 and 4 when both equality and inequality constraints are present. The details are given in [1].

ACKNOWLEDGMENT

Mr. Andreassen's share of the work was done while he was in receipt of a Council of Europe Higher Education Scholarship.

REFERENCES

1. D. O. ANDREASSEN, The solution of Chebyshev approximation problems, with and without constraints, Ph. D. Thesis, University of Dundee, 1976.
2. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
3. A. R. CURTIS AND M. J. D. POWELL, Necessary conditions for a minimax approximation, *Comput. J.* **8** (1966), 358–361.
4. H. G. EGGLESTON, "Convexity," Cambridge Univ. Press, Cambridge, 1958.
5. G. A. GISLASON AND H. L. LOEB, Nonlinear Tchebycheff approximation with constraints, *J. Approximation Theory* **6** (1972), 291–300.
6. K. H. HOFFMANN, Zur Theorie der nichtlinearen Tschebyscheff-Approximation mit Nebenbedingungen, *Numer. Math.* **14** (1970), 24–41.

7. K. H. HOFFMANN, Zur Theorie der nichtlinearen Tschebyscheff-Approximation mit Nebenbedingungen: Anwendungen, *Numer. Math.* **14** (1970), 383–393.
8. P. J. LAURENT, "Approximation et Optimisation," Hermann, Paris, 1972.
9. J. T. LEWIS, Approximation with convex constraints, *SIAM Rev.* **15** (1973), 193–217.
10. G. D. TAYLOR, Uniform Approximation with Side Conditions, *in* "Approximation Theory," pp. 495–503, Proceedings of an International Symposium, G. G. Lorentz, Ed., Academic Press, New York/London, 1973.
11. G. A. WATSON, A method for calculating best nonlinear Chebyshev approximations, *J. Inst. Math. Appl.*, to appear.